FURTHER ILLUSTRATION OF THE USE OF THE FROBENIUS-SCHWINGER-DYSON EQUATIONS

OLIVIER DE MIRLEAU

ABSTRACT. The Frobenius-Schwinger-Dyson equations are a rather high-brow abstract nonsense type of equations describing n-point functions of arbitrarily high composite insertions. It is not clear how to solve or even find approximate solutions of these equations in general, but they are worth investigating because (a certain preferred type of) renormalization of composite insertions has been performed in advance: it just remains to find solutions given an action and renormalization conditions. Earlier work in this field involved only Gaussian actions or variable transformations thereof. In this work we illustrate the use of Frobenius-Schwinger-Dyson at a less obviously trivial level, that of the Thirring model.

Keywords: Schwinger-Dyson equation, Frobenius Algebras, Thirring model.
1991 MSC: 81 Q 40, 81 R 99, 81 T 40.

CONTENTS

1. INTRODUCTION AND OVERVIEW

In a previous paper [5], we introduced the notion of Frobenius-Schwinger-Dyson equations associated to an action. These equations which we will remind the in next section are equations for renormalized expectation values $\langle\mathcal{O}_1, .., \mathcal{O}_n\rangle_r$ of possibly composite insertions $\mathcal{O}_i$.

Under the assumption that these expectation values are obtained as

$$\langle\mathcal{O}_1, .., \mathcal{O}_n\rangle_r := \langle R(\mathcal{O}_1)..R(\mathcal{O}_n)\rangle,$$

where $\langle.\rangle$ is a solution of the Schwinger-Dyson equation associated to the action S, and R is a regulator of composite insertions, and if one furthermore assumes a compatibility between R and S, one can derive equations for the renormalized expectation values.

This work was financially supported by "Samenwerkingsverband FOM/SWON Mathematische Fysica", under grant no. MF92.3.

To get an idea of what this compatibility is, note the following: In case S is quadratic, R is typically to be though of as the normal ordering N_S associated to S, but if S is not quadratic then R is *not* to be thought of as normal ordering N_0 of some quadratic action S_0, but rather as an operation which is directly related to S.Now it is not clear how to generalize normal ordering to arbitrary non-quadratic actions S, so that one is forced to think about what it means to say that R is a regulator of composite insertions directly related to S. The mere requirement that infinite-dimensional calculations give finite results would reduce the theory to triviality since we might in that case just as well take $R = 0$. In [5] it was noted that normal ordering N_S for quadratic actions (or in fact a larger class for which it *is* clear how to generalize normal ordering) satisfies a compatibility condition with S such that

1. The compatibility condition can be generalized to arbitrary action S. (This is fortunate).
2. The compatibility condition does not determine N_S once S is given. (Unfortunate).

Forgetting about point two for a moment, we were then led to *define* regulators as those operations R that satisfy the compatibility condition, which in turn implies identities for the corresponding renormalized expectation values, which we called the Frobenius-Schwinger-Dyson equations. These equations seemed so natural that it was more or less suggested that computing functional integrals with composite insertions and fixed action is *nothing but* solving the FSD equations, although that point was only illustrated with Gaussian integrals and variable transformations thereof.

Finally, coming back to point two one notes that Frobenius-Schwinger-Dyson equations associated to an action can have multiple solutions if they are not supplemented with extra renormalization conditions.

In this article we will push the illustration of the FSD equations a little further by using them to derive the expression for renormalized $2n$-point functions

$$\langle\bar{\psi}^{A_1}(x_1), \bar{\psi}^{A_2}(x_2), \bar{\psi}^{A_3}(x_3), ..., \psi^{B_1}(y_1), \psi^{B_2}(y_2), \psi^{B_3}(y_3), ...\rangle_r$$

first found by Johnson [1] for a two-dimensional Fermion field with the Thirring action. To fix them, we will have to impose some renormalization conditions, which we will choose in such a way as to remain as close as possible to Johnson's expressions. What Johnson's method amounts to in our setting is to guess the form of renormalization conditions that might allow for solutions of the FSD equations, and subsequently fixing free parameters in this guess just by the very imposition of these FSD equations.

2. Reminder on the Frobenius-Schwinger-Dyson equation

We recall the following from [5]:

Definition 2.1. *Let A be a symmetric associative algebra with unit and typical elements f, g, let $L := Der(A)$ with typical elements X, Y and corresponding operations fg, fX, Xf and $[X, Y]$. By a renormalized structure on A we mean the datum of $S \in A$, together with extra multiplications $f._r g$, $f._r X$, $X._r f$ and $[X,_r Y]$, also satisfying associativity, Jacobi identity etc., such that*

1. *The $._r$-operations induce $L = Der(A, ._r)$, i.e. the derivations of the multiplication $(f, g) \mapsto f._r g$ are exactly given by the operations $f \mapsto X._r f$ as $X \in L$.*
2. *Both structures have the same unit: $1f = f = 1._r f$.*

3. *The two algebraic structures and the action S are compatible in the sense that:*

$$X._r(YS) - Y._r(XS) = [X,_r Y]S,$$

$$(f._rX)S = f._r(XS) - X._rf.$$

The algebra A together with the renormalized structure will be called a renormalized volume manifold. For symmetric linear maps $\langle ,,,\rangle_r : A^{\otimes n} \to \mathbb{R}$ we will be interested in the following properties:

1. $\langle .,.\rangle_r : A^{\otimes 2} \to \mathbb{R}$ *is a positive non degenerate form. (Positivity).*
2. $\langle \mathcal{O}_1, \mathcal{O}_2, ..., \mathcal{O}_n\rangle_r = \langle \mathcal{O}_1._r\mathcal{O}_2, .., \mathcal{O}_n\rangle_r$, *(Frobenius).* [1]
3. $\langle XS, \mathcal{O}_1, ..., \mathcal{O}_n\rangle_r = \sum_{i=1}^n \langle \mathcal{O}_1, .., X._r\mathcal{O}_i, .., \mathcal{O}_n\rangle_r$. *(Schwinger-Dyson).*

By substituting $S/\hbar$ for S and taking the limit $\hbar \to 0$, we see that at $\hbar = 0$, we may take both algebraic structures to be equal. In what follows, we will take the usual multiplication to have priority over the renormalized one, i.e. $f._rgh$ means $f._r(gh)$.

In this article we will concern ourselves with the case where A is the algebra of words in odd symbols $\psi^A(x), \bar{\psi}^A(x)$ and their derivatives, and S is given by the Thirring action.

3. Reminder on chiral currents

3.1.1. *Chiral currents.* By a chiral current we shall understand a current J which is divergenceless and rotationfree if the Euler-Lagrange equations are satisfied[2]. More precisely, currents such that there are functional vector fields Y and $Y_{\mu\nu}$ such that

$$\partial_\mu J^\mu = YS,$$

$$\partial_\mu J_\nu - \partial_\nu J_\mu = Y_{\mu\nu}S.$$

In the context of renormalized volume manifolds, one may now define $[J^\mu \triangleright_r .] : A \to A$ by the following properties:

$$\partial_\mu [J^\mu \triangleright_r \mathcal{O}] := Y._r\mathcal{O},$$

$$\partial_\mu [J_\nu \triangleright_r \mathcal{O}] - \partial_\nu [J_\mu \triangleright_r \mathcal{O}] := Y_{\mu\nu}._r\mathcal{O}.$$

$$[J^\mu(\infty), \triangleright_r \mathcal{O}] := 0.$$

Note that this differential equation for $[J^\mu \triangleright_r .]$ can be solved rather explicitly in terms of an integral expression of the right hand side.

We now see that if $\langle J^\mu(\infty), \mathcal{O}\rangle_r = 0$ then the chiral currents satisfy the identity

$$\langle J^\mu, \mathcal{O}\rangle_r = \langle [J^\mu \triangleright_r \mathcal{O}]\rangle_r.$$

Such an identity may in turn be used to produce differential equations for n-point functions in case certain identities are known for the $._r$ operations. Such identities may be known in the following cases:

1. Everything was already solved by some other method, say because S was Gaussian.
2. One guesses properties of $._r$.

[1] A Frobenius algebra is a symmetric associative algebra with metric such that $(a, bc) = (ab, c)$.

[2] In two dimensions this reduces to the more common meaning of a current $J = *J_\mu dx^\mu = fdz + gd\bar{z}$ being chiral iff f is holomorphic and g antiholomorphic.

3.1.2. *Example 1.* As a first example consider the action $S(\phi) = \int \partial_\mu \phi \partial^\mu \phi d^D x/2$ for scalar fields $\phi : \mathbb{R}^D \to \mathbb{R}$, and renormalization condition

$$\mathcal{O}_1 \cdot_r \frac{\delta}{\delta\phi(x)} \mathcal{O}_2 = \mathcal{O}_1 \frac{\delta}{\delta\phi(x)} \cdot_r \mathcal{O}_2.$$

This renormalization condition was discussed in more detail in [5]: The FSD equations for non degenerate Gaussian integrals subjected to this condition allow for a unique solution usually referred to as *the* solution for this action. The solution of the $\cdot_r$ multiplication in terms of the usual one can for this renormalization condition and action S be inductively expressed as follows (see appendix, proof 1):

$$(\frac{\delta S}{\delta\phi(x)} \mathcal{O}_1) \cdot_r \mathcal{O}_2 = \frac{\delta S}{\delta\phi(x)} (\mathcal{O}_1 \cdot_r \mathcal{O}_2) + \mathcal{O}_1 \cdot_r \frac{\delta}{\delta\phi(x)} \mathcal{O}_2.$$

Now the action above also happens to have a chiral current $j_\mu := \partial_\mu \phi$, by taking $Y^x := -\delta/\delta\phi(x)$, and $Y_{\mu\nu} := 0$. The expression $V_p(x) := \exp(p\phi(x))$ then has the property that its derivative can be algebraically expressed in terms of j_μ and V_p itself:

$$\partial_\mu V_p = p j_\mu V_p.$$

The properties of $\cdot_r$ that are needed to derive a differential equation for n-point functions of the V_p's are

1. $Y^x \cdot_r V_p(y) = Y^x V_p(y)$,
2. $j_\mu(x) V_p(x) = \lim_{y \to x} j_\mu(y) \cdot_r V_p(x) - [j_\mu(y) \triangleright_r V_p(x)]$.

Indeed, the first equation above follows directly from the main renormalization condition; For the second equation we use $j_\mu(x)V_p(x) = \overline{\lim}_{\epsilon \to 0} j_\mu(y) V_p(x)$, and use the Gaussian recursion relation for the $\cdot_r$ product.

The way these two equations imply a differential equation for n-point functions is as follows:

$$\begin{aligned} \partial_{x^\mu} \langle V_p(x), \mathcal{O}\rangle_r &= p\langle j_\mu(x) V_p(x), \mathcal{O}\rangle_r = p \lim_{y \to x} \langle (j_\mu(y) \cdot_r V_p(x) - [j_\mu(y) \triangleright_r V_p(x)]), \mathcal{O}\rangle_r \\ &= p \lim_{y \to x} \langle j_\mu(y), V_p(x) \cdot_r \mathcal{O}\rangle_r - \langle [j_\mu(y) \triangleright_r V_p(x)], \mathcal{O}\rangle_r = \lim_{y \to x} \langle V_p(x), [j_\mu(y) \triangleright_r \mathcal{O}]\rangle_r \\ &= \langle V_p(x), [j_\mu(x) \triangleright_r \mathcal{O}]\rangle_r \end{aligned}$$

3.1.3. *Example 2:* Next consider a slightly more complicated case, that of Fermionic spinors ψ and $\bar\psi$, with Dirac action $S = \int \bar\psi(\gamma^\mu \partial_\mu - im)\psi d^D x$, together with the Fermionic analogue of the renormalization condition of the previous example. This case is just as solvable as the previous one since it is also Gaussian. The current $j_\mu := \bar\psi \gamma_\mu \psi$ is chiral in case we have $m = 0$ and $D = 2$: Indeed defining the functional vector fields

$$Y^x := (\bar\psi^A(x) \frac{\delta}{\delta\bar\psi^A(x)} - \psi^A(x) \frac{\delta}{\delta\psi^A(x)}),$$

$$Y_{\mu\nu} := \frac{1}{2}(\psi^A [\gamma_\mu, \gamma_\nu]_A{}^B \frac{\delta}{\delta\psi^B} - \bar\psi^A [\gamma_\mu, \gamma_\nu]_A{}^B \frac{\delta}{\delta\bar\psi^B}),$$

we have

$$\partial_\mu j^\mu = YS,$$

$$\partial_\mu j_\nu - \partial_\nu j_\mu - Y_{\mu\nu} S = \frac{1}{2}\{\partial_\sigma \bar\psi \Gamma_{\mu\sigma\nu} \psi - \bar\psi \Gamma_{\mu\sigma\nu} \partial_\sigma \psi + 2im\bar\psi[\gamma_\mu, \gamma_\nu]\psi\},$$

$$\textit{where } \Gamma_{\mu\sigma\nu} := \gamma_\mu \gamma_\sigma \gamma_\nu - \gamma_\nu \gamma_\sigma \gamma_\mu,$$

which is zero in $D = 2$. Assuming that $m = 0$ and $D = 2$, one may again derive properties of $\cdot_r$ that imply a differential equation, for n-point functions ψ and $\bar\psi$'s this time: Setting $\varepsilon_{\mu\nu}\tilde{Y} := Y_{\mu\nu}$, where $dx^\mu \wedge dx^\nu =: \varepsilon^{\mu\nu} dx^1 \wedge dx^2$, and using the notion

$$\overline{\lim}_{\epsilon\to(0,0)\in\mathbb{R}^2}\, f(\epsilon) := \lim_{\epsilon\to 0} \frac{1}{4}\sum_{i=0}^{3} f(Rot_{i\times 90^o}(\epsilon)),$$

so that in particular $\overline{\lim}_{\epsilon\to 0}\, \epsilon_\alpha f(|\epsilon|^2) = 0$, and $\overline{\lim}_{\epsilon\to 0}\, \frac{\epsilon^\alpha\epsilon^\beta}{|\epsilon|^2} = \frac{g^{\alpha\beta}}{2}$, we have:

1. $Y\cdot_r\psi = Y\psi$; $Y\cdot_r\bar\psi = Y\bar\psi$; $\tilde{Y}\cdot_r\psi = \tilde{Y}\psi$; $\tilde{Y}\cdot_r\bar\psi = \tilde{Y}\bar\psi$
2. $\overline{\lim}_{\epsilon\to 0}\{j_\mu(x+\epsilon)\cdot_r\psi^A(x) - [j_\mu(x+\epsilon) \triangleright_r \psi^A(x)] - j_\mu(x+\epsilon)\psi^A(x)\}$
$$= \frac{-1}{4\pi}(\gamma^\nu\gamma_\mu)^A{}_B\partial_\nu\psi^B(x).$$
(See appendix, proof 2).
3. $\frac{\delta}{\delta\psi^A(x)}\cdot_r\psi^B(y) = \delta_A^B\delta(x-y)$.
4. $j^\mu(x) = \overline{\lim}_{\epsilon\to 0}\, \bar\psi^A(x)\cdot_r(\gamma_\mu)_{AB}\psi^B(x+\epsilon)$.

That these properties by themselves imply a differential equation for the $2n$-point functions will be shown in more generality in the next section, from which the present case is recovered by setting $\lambda := 0$.

4. The Thirring model with Johnson's renormalization conditions

In the associative algebra on odd symbols $\psi^A(x)$ and $\bar\psi^A(x)$ where $x \in \mathbb{R}^2$ and ^{A} is an index in a $2D$ representation space for the Clifford algebra of $\mathbb{R}^2$ with $\gamma^\mu_{AB} = \gamma^\mu_{BA}$, Thirring's Lagrangian is defined as $\mathcal{L}(\bar\psi,\psi) := \bar\psi\gamma^\mu\partial_\mu\psi + \frac{\lambda}{2}j^\mu j_\mu$, where $j^\mu := \bar\psi\gamma^\mu\psi$. Consequently, we have

1. $\frac{\delta\mathcal{L}}{\delta\bar\psi^A} = (\gamma^\mu)_{AB}\partial_\mu\psi^B + \lambda(\gamma^\mu)_{AB}j_\mu\psi^B$.
2. $\frac{\delta\mathcal{L}}{\delta\psi^A} = \partial_\mu\bar\psi^B(\gamma^\mu)_{BA} - \lambda(\gamma^\mu)_{AB}j_\mu\bar\psi^B$.
3. $\partial_\mu j^\mu = YS$
4. $\partial_\mu j_\nu - \partial_\nu j_\mu = Y_{\mu\nu}S$.

I.e. although the Lagrangian is not quadratic any more, the current j_μ remains chiral even when $\lambda \neq 0$, which basically makes a solution of the Thirring model possible.

Next, we need renormalization conditions to fix the renormalized $2n$-point functions. Now it is not evident what kind of conditions can be imposed such that solutions of the Frobenius-Schwinger-Dyson equations satisfying them actually exist. This general problem will remain unaddressed in this paper: We will restrict ourselves to deriving a number of Johnson's results [1] by using the Frobenius-Schwinger-Dyson equations: In our language, Johnson made a smart guess about the form of a number of renormalization conditions, by introducing some undetermined parameters $a, \tilde{a}$ in the conditions as they hold in the Gaussian $\lambda = 0$ case:

1. $Y\cdot_r\psi = aY\psi$; $Y\cdot_r\bar\psi = aY\bar\psi$; $\tilde{Y}\cdot_r\psi = \tilde{a}\tilde{Y}\psi$; $\tilde{Y}\cdot_r\bar\psi = \tilde{a}\tilde{Y}\bar\psi$, see [1, formulae 14,15].
2. $\overline{\lim}_{\epsilon\to 0}\{j_\mu(x+\epsilon)\cdot_r\psi^A(x) - [j_\mu(x+\epsilon) \triangleright_r \psi^A(x)] - j_\mu(x+\epsilon)\psi^A(x)\}$
$$= \frac{-b}{4\pi}(\gamma^\nu\gamma_\mu)^A{}_B\partial_\nu\psi^B(x).$$
3. $x \neq y \;\Rightarrow\; \frac{\delta}{\delta\psi^A(x)}\cdot_r\psi^B(y) = 0$[3].

[3]We will not address the question whether this condition can be consistently extended to be a distribution for all values of x and y including $x = y$. The price we pay is that solutions of the differential equations that follow need not be unique.

4. $j^\mu(x) = \overline{\lim}_{\epsilon\to 0}\, U_{\lambda,a,\tilde a}(\epsilon^2)\bar\psi^A(x)._r(\gamma_\mu)_{AB}\psi^B(x+\epsilon)$, see [1, formula 29] and [7, formula 4.71],

where $U_{\lambda,a,\tilde a}$ is defined by $\langle\bar\psi^A(\epsilon),\psi^B(0)\rangle_r^{\lambda,a,\tilde a} = U^{-1}_{\lambda,a,\tilde a}(\epsilon^2)\langle\bar\psi^A(\epsilon),\psi^B(0)\rangle_r^{\lambda=0,a=1,\tilde a=1}$, which as we will see exists. We see that the Gaussian properties with canonical renormalization condition are recovered by setting $a=\tilde a = b = 1$. The parameter b does not occur in Johnson's work, but we will need condition 2 to make everything work. It will not enter the final answer because the b-term will be multiplied by γ^μ, and since in two dimensions $\gamma^\mu\gamma_\nu\gamma_\mu = 0$.

Repeating the procedure that we have seen before to produce differential equations, we see that the defining differential equation for $[j_\mu \triangleright_r .]$ acting on $\psi,\bar\psi$ is solved by

$$[j^\mu(x)\triangleright_r \psi^A(y)] = (-af^\mu(x-y)\delta^A_B + \tilde a\varepsilon^{\mu\nu}f_\nu(x-y)\bar\gamma^A{}_B)\psi^B(y),$$
$$[j^\mu(x)\triangleright_r \bar\psi^A(y)] = (af^\mu(x-y)\delta^A_B - \tilde a\varepsilon^{\mu\nu}f_\nu(x-y)\bar\gamma^A{}_B)\bar\psi^B(y),$$

where $\bar\gamma := \gamma^1\gamma^2$, $f(x) := (\ln|x|)/2\pi$, and $f_\mu(x) := \partial_\mu f(x)$. For example:

$$\partial_\mu[j^\mu(x)\triangleright_r\psi^A(y)] = -a\partial_\mu f^\mu(x-y)\delta^A_B\psi^B(y) = -a\delta(x-y)\delta^A_B\psi^B(y) = aY^x\psi^A(y) = Y^x._r\psi^A(y).$$

The differential equation that now holds is the following if $\mathcal{O}$ is a repeated $._r$-product of $\psi(y_i),\bar\psi(y_i)$'s where $x\neq y_i$:

$$\langle(\gamma^\mu)^A{}_B\partial_\mu\psi^B(x),\mathcal{O}\rangle_r = -\lambda(\gamma^\mu)^A{}_B\langle\psi^B(x),[j_\mu(x)\triangleright_r\mathcal{O}]\rangle_r.$$

Proof

$$LHS = \langle\frac{\delta\mathcal{L}}{\delta\bar\psi^A(x)} - \lambda j_\mu(x)(\gamma^\mu\psi)^A(x),\mathcal{O}\rangle = -\lambda(\gamma^\mu)^A{}_B\langle j_\mu(x)\psi^B(x),\mathcal{O}\rangle$$
$$= -\lambda(\gamma^\mu)^A{}_B\overline{\lim}_{\epsilon\to 0}\langle j_\mu(x+\epsilon)._r\psi^B(x) - [j_\mu(x+\epsilon)\triangleright_r\psi^B(x)] + \frac{b}{4\pi}(\gamma^\nu\gamma_\mu)^B{}_C\partial_\nu\psi^C(x),\mathcal{O}\rangle$$
$$= -\lambda(\gamma^\mu)^A{}_B\langle\psi^B(x),[j_\mu(x)\triangleright_r\mathcal{O}]\rangle - \frac{\lambda b}{4\pi}(\gamma^\mu\gamma^\nu\gamma_\mu)^A{}_B\langle\partial_\nu\psi^C,\mathcal{O}\rangle.$$

□

Up to now, renormalization condition 4 has not been used. We will prove in the next section that it implies the relations first found by Johnson

$$a = \frac{1}{1-\lambda/2\pi},\quad \tilde a = \frac{1}{1+\lambda/2\pi}.$$

5. Free field realization, restriction of the parameters $a,\tilde a$.

The differential equation satisfied by the expectation values of the Thirring model can be solved by expectation values of special integrands of a Gaussian weight: Indeed, define the following Gaussian contractions [4] for two Bosonic fields ϕ_1 and ϕ_2 and Dirac fields χ and $\bar\chi$:

$$[\phi_i(x)\triangleright\phi_j(y)] := -\delta_{ij}f(x-y),$$
$$[\chi\triangleright\chi] := [\bar\chi\triangleright\bar\chi] := 0,$$
$$[\chi^A(x)\triangleright\bar\chi^B(y)] := (\gamma^\mu)^{AB}f_\mu(x-y) =: [\bar\chi^A(x)\triangleright\chi^B(y)].$$

Following [2, formula IV.5], let us define

$$\Psi^A(x) := (e^{-\lambda k_1\phi_1(x)-\lambda k_2\phi_2(x)\bar\gamma})^A{}_B\chi^B(x),$$
$$\bar\Psi^A(x) := (e^{+\lambda k_1\phi_1(x)+\lambda k_2\phi_2(x)\bar\gamma})^A{}_B\bar\chi^B(x),$$

for yet to be fixed numbers k_1, k_2. Making the Dirac operator $\gamma^\mu \partial_\mu$ act on these expressions will bring down the matrix $\bar{\gamma}$, and using that $\gamma^\mu \bar{\gamma} = \varepsilon^{\mu\nu}\gamma_\nu$, we become interested in

$$j^\mu_\pm := k_1 \partial^\mu \phi_1 \pm k_2 \varepsilon^{\mu\nu}\partial_\nu \phi_2$$

By Gaussian Wick calculus we then have:

$$\begin{aligned}
&[j^\mu_-(x) \triangleright \Psi^A(y)] = \{\lambda k_1^2 f^\mu(x-y)\delta^A_B - \lambda k_2^2 \varepsilon^{\mu\nu} f_\nu(x-y)\bar{\gamma}^A{}_B\}\Psi^B(y),\\
&[j^\mu_-(x) \triangleright \bar{\Psi}^A(y)] = \{-\lambda k_1^2 f^\mu(x-y)\delta^A_B + \lambda k_2^2 \varepsilon^{\mu\nu} f_\nu(x-y)\bar{\gamma}^A{}_B\}\bar{\Psi}^B(y),\\
&\frac{\delta \mathcal{O}}{\delta \bar{\psi}(x)} = 0 \Rightarrow (\gamma_\mu)^A{}_B \langle \partial_\mu \Psi^B(x), \mathcal{O}\rangle_r = -\lambda (\gamma_\mu)^A{}_B \langle \Psi^B(x), [j^\mu_-(x) \triangleright \mathcal{O}]\rangle_r.
\end{aligned}$$

The last differential equation is exactly the one derived using Johnson's renormalization conditions, provided the action $[j^\mu_- \triangleright .]$ equals the action of j^μ, which happens if we have $-\lambda k_1^2 = a$ and $-\lambda k_2^2 = \tilde{a}$. Consequently the n-point functions of the Ψ's above with these values of k_i satisfy the differential equation for the ψ's of the Thirring n-point functions. In particular the two-point function can be computed, leading to

$$\langle \bar{\psi}^A(\epsilon), \psi^B(0)\rangle_r^{\lambda,a,\tilde{a}} = e^{\lambda(a-\tilde{a})\langle \phi(0)\phi(\epsilon)\rangle} \langle \bar{\psi}^A(\epsilon), \psi^B(0)\rangle_r^{\lambda=0,a=1,\tilde{a}=1},$$

so that $U^{-1}_{\lambda,a,\tilde{a}}(\epsilon^2) = e^{\lambda(a-\tilde{a})\langle\phi(0)\phi(\epsilon)\rangle}$. Furthermore, in view of renormalization condition 4 the Thirring expectation values of $j_\mu(x)$ are reproduced by the following expression in terms of the Gaussian fields:

$$J^\mu(x) := \overline{\lim}_{\epsilon\to 0}\, U_{\lambda,a,\tilde{a}}(\epsilon^2)\bar{\Psi}^A(x)\cdot_r (\gamma_\mu)_{AB}\Psi^B(x+\epsilon).$$

But using Gaussian Wick calculus, one may prove (see appendix, proof 3) that

$$J^\mu = \bar{\Psi}^A(\gamma^\mu)_{AB}\Psi^B + \frac{\lambda}{2\pi} j^\mu_+ = \bar{\chi}^A(\gamma^\mu)_{AB}\chi^B + \frac{\lambda}{2\pi} j^\mu_+ =: j^\mu_\chi + \frac{\lambda}{2\pi} j^\mu_+.$$

Since $\langle j^\mu, \mathcal{O}\rangle_r = \langle [j^\mu \triangleright_r \mathcal{O}]\rangle_r$, and comparing $[j^\mu \triangleright_r \psi^A]$ with $[j^\mu_\chi + \frac{\lambda}{2\pi} j^\mu_+ \triangleright \Psi^A]$, we see that the following two expressions must be equal:

1. $(-a f^\mu(x-y)\delta^A_B + \tilde{a}\varepsilon^{\mu\nu} f_\nu(x-y)\bar{\gamma}^A{}_B)\psi^B(y)$,
2. $(-(1+\frac{\lambda a}{2\pi})f^\mu(x-y)\delta^A_B + (1-\frac{\lambda\tilde{a}}{2\pi})\varepsilon^{\mu\nu} f_\nu(x-y)\bar{\gamma}^A{}_B)\psi^B(y)$,

giving that $a = \frac{1}{1-\lambda/2\pi}$ $\tilde{a} = \frac{1}{1+\lambda/2\pi}$ as promised.

6. Partial vs. Total renormalization conditions.

By partial renormalization conditions we shall understand renormalization conditions which together with the FSD equation do not completely determine the $\cdot_r$ operations, as opposed to total renormalization conditions which will be those which do fix the $\cdot_r$'s: For example $f\partial_i \cdot_r g = f \cdot_r \partial_i g$ is a total renormalization condition for Gaussian actions.

Johnson's renormalization conditions for the Thirring model seem to be only partial, since when restricted to the Gaussian $\lambda = 0$ case, Johnson's conditions are obviously weaker than $f\partial_i \cdot_r g = f \cdot_r \partial_i g$.

One may ask if there is a natural way to strengthen Johnson's renormalization conditions so that they also determine the operator product or expectation values of higher order insertions. A simple try at guessing such an extension is to first extend the free field realization $F : \psi \mapsto \exp(-\lambda k_1 \phi_1 - \lambda k_2 \phi_2 \bar{\gamma})\psi_0$ by $F(ab) := F(a)F(b)$, and then trying to pull-back the Gaussian $\cdot_r$ operations by F. This however fails because $F(a)\cdot_r F(b)$ is not in the image of F for all a and b. Another try could be to use the bosonization procedure as a free field realization. The same problems as above might in that case occur although we didn't check that.

7. Conclusion

What we have seen is that the Frobenius-Schwinger-Dyson equations can reproduce known expressions for renormalized expectation values, the advantage being that there was no need of explicitly regularizing and renormalizing the fields. It remains to be proved that there actually exists a solution of the FSD equations for the Thirring model.

It would seem that the same method will apply for example to the Wess-Zumino-Witten model, since the derivation [3] of differential equations for n-point functions in that case is not different from Johnson's derivation: Arbitrary multiplicative parameters $\kappa = \bar{\kappa}$ have been introduced [3, formula 3.1], and consistency with renormalization conditions on higher composite operators fixes the value of κ in terms of the integer coupling constant k of the model [3, formula 3.11].

¿From a more fundamental point of view it remains an open question, even in finite dimensions, whether there is an algorithm which given an action and renormalization conditions will produce approximations of solutions of the FSD equations up to arbitrarily high precision. Unlike approximation schemes for positive solutions of the usual Schwinger-Dyson equation, which are just the approximation schemes for integration, an approximation scheme for solutions of the FSD equations seems difficult to construct given that the number of solutions will in general depend on the renormalization conditions.

8. Acknowledgements

Thanks are due to R. H. Dijkgraaf, H. G. B. Pijls and C. D. D. Neumann for fruitful discussions.

Appendix A. Proofs

1. The renormalization condition $f\cdot_r\partial_i g = f\partial_i\cdot_r g$ implies $(f\cdot_r g)\partial_i = f\cdot_r(g\partial_i)$ and $((\partial_i S)\mathcal{O}_1)\cdot_r\mathcal{O}_2 = (\partial_i S)(\mathcal{O}_1\cdot_r\mathcal{O}_2) + \mathcal{O}_1\cdot_r\partial_i\mathcal{O}_2$:

   Proof

   To prove the first formula, we have to show that both sides of the equation have the same $\cdot_r$ action:

$$[(f\cdot_r g)\partial]\cdot_r h = (f\cdot_r g)\cdot_r\partial h = f\cdot_r(g\cdot_r\partial h) = f\cdot_r(g\partial\cdot_r h) = (f\cdot_r g\partial)\cdot_r h.$$

   The second formula now follows from the first as follows: In the formula $f\cdot_r XS - X\cdot_r f = (f\cdot_r X)S$ that holds by definition of renormalized volume manifolds, take $f := \mathcal{O}_2$, and $X := \mathcal{O}_1\partial_i$. This gives

$$\begin{aligned}\mathcal{O}_2\cdot_r(\mathcal{O}_1\partial_i S) - \mathcal{O}_1\cdot_r\partial_i\mathcal{O}_2 &= \mathcal{O}_2\cdot_r(\mathcal{O}_1\partial_i S) - \mathcal{O}_1\partial_i\cdot_r\mathcal{O}_2 \\ &= f\cdot_r XS - X\cdot_r f = (f\cdot_r X)S = (\mathcal{O}_2\cdot_r\mathcal{O}_1\partial_i)S\end{aligned}$$

   By the first formula, the last term equals $(\mathcal{O}_2\cdot_r\mathcal{O}_1)\partial_i S$.

   □

2. For Gaussian Dirac fields we have:

$$\begin{aligned}&\overline{\lim}_{\epsilon\to 0}\{j_\mu(x+\epsilon)\cdot_r\psi^A(x) - [j_\mu(x+\epsilon)\triangleright_r\psi^A(x)] - j_\mu(x+\epsilon)\psi^A(x)\} \\ &\qquad = \frac{-1}{4\pi}(\gamma^\nu\gamma_\mu)^A{}_B\partial_\nu\psi^B(x).\end{aligned}$$

   Proof

   Indeed, the left hand side is the limit of

$$[\psi^A(x)\triangleright j_\mu(x+\epsilon)] - [j_\mu(x+\epsilon)\triangleright_r\psi^A(x)]$$

$$= [\psi^A(x) \triangleright \bar{\psi}^B(x+\epsilon)](\gamma_\mu)_{BC}\psi^C(x+\epsilon) + f_\nu(\epsilon)(\gamma^\nu\gamma_\mu)_{AC}\psi^C(x)$$

$$= -(\gamma^\rho\gamma_\mu)_{AC} f_\rho(\epsilon)\{\psi^C(x+\epsilon) - \psi^C(x)\} = -(\gamma^\rho\gamma_\mu)_{AC}\frac{\epsilon_\rho\epsilon_\nu}{2\pi\epsilon^2}\partial^\nu\psi^C(x) + O(\epsilon),$$

which in the limit approaches right hand side.

□

3. The different currents introduced in section 5 satisfy the following identity:

$$J^\mu = \bar{\Psi}^A(\gamma^\mu)_{AB}\Psi^B + \frac{\lambda}{2\pi}j^\mu_+ = \bar{\chi}^A(\gamma^\mu)_{AB}\chi^B + \frac{\lambda}{2\pi}j^\mu_+ =: j^\mu_\chi + \frac{\lambda}{2\pi}j^\mu_+.$$

Proof
First by expanding roughly speaking $\Psi = e^{-(\dots)}\chi$ and $\bar{\Psi} = e^{+(\dots)}\bar{\chi}$, so that $\bar{\Psi}\cdot_r\Psi = (e^+\bar{\chi})\cdot_r(e^-\chi) = \{e^+\}\cdot_r\{e^-\}(\bar{\chi}\chi + \langle\bar{\chi}\chi\rangle)$, and using the matrix version of $e^{px}\cdot_r e^{qx} = e^{pq}e^{(p+q)x}$, we have:

$$e^{-\lambda a\langle\phi(x)\phi(y)\rangle - \lambda\tilde{a}\langle\phi(x)\phi(y)\rangle\bar{\gamma}_x\bar{\gamma}_y}(\bar{\Psi}(x)\cdot_r\Psi(y))$$

$$= \bar{\Psi}(x)\Psi(y) + e^{\sqrt{-\lambda a}(\phi_1(x)-\phi_1(y)) + \sqrt{-\lambda\tilde{a}}(\phi_2(x)\bar{\gamma}_x - \phi_2(y)\bar{\gamma}_y)}\langle\bar{\chi}(x)\chi(y)\rangle,$$

where $\bar{\gamma}_x$ acts on $\bar{\chi}(x)$, i.e. $(\bar{\gamma}_x\bar{\chi}(x)\chi(y))^{AB} = (\bar{\gamma}^A{}_C\bar{\chi}^C(x)\chi^B(y))$. ¿From this formula one deduces the identity among currents by taking the limit $y \to x$:

$$j^\mu_\chi(x) = \bar{\Psi}^A(x)\gamma^\mu_{AB}\Psi^B(x) = \overline{\lim}_{\epsilon\to 0}\ \bar{\Psi}^A(x)\gamma^\mu_{AB}\Psi^B(x+\epsilon =: y) = I - II,$$

where with $B_\epsilon := \langle\phi(0)\phi(\epsilon)\rangle$:

$$I = \overline{\lim}_{\epsilon\to 0}(e^{-\lambda a B_\epsilon - \lambda\tilde{a}B_\epsilon\bar{\gamma}_x\bar{\gamma}_y})^A{}_C{}^B{}_D\bar{\Psi}^C(x)\cdot_r\Psi^D(y)\gamma^\mu_{AB}$$

$$= \overline{\lim}_{\epsilon\to 0}e^{-\lambda(a-\tilde{a})B_\epsilon}\bar{\Psi}^C(x)\cdot_r\gamma^\mu_{CD}\Psi^D(y) = J^\mu(x)$$

$$II = \overline{\lim}_{\epsilon\to 0}\ [e^{\sqrt{-\lambda a}(\phi_1(x)-\phi_1(y)) + \sqrt{-\lambda\tilde{a}}(\phi_2(x)\bar{\gamma}_x - \phi_2(y)\bar{\gamma}_y)}]^A{}_C{}^B{}_D(\gamma^\rho)^{CD}f_\rho(x-y)(\gamma^\mu)_{AB}$$

$$= \overline{\lim}_{\epsilon\to 0}\ e^{\sqrt{-\lambda a}(\phi_1(x)-\phi_1(y))}f_\rho(x-y)Tr(e^{\sqrt{-\lambda\tilde{a}}(\phi_2(x)-\phi_2(y))\bar{\gamma}}\gamma^\rho\gamma^\mu)$$

$$= \overline{\lim}_{\epsilon\to 0}\ e^{\sqrt{-\lambda a}(\phi_1(x)-\phi_1(y))}f_\rho(x-y)$$

$$\times\{2g^{\rho\mu}\cos[\sqrt{-\lambda a}(\phi_1(x)-\phi_1(y))] - 2\varepsilon^{\rho\mu}\sin[\sqrt{-\lambda a}(\phi_1(x)-\phi_1(y))]\}$$

$$= \overline{\lim}_{\epsilon\to 0}\ [1 - \sqrt{-\lambda a}\epsilon^\kappa\partial_\kappa\phi_1(x) + O(\epsilon^2)]\frac{-\epsilon_\rho}{2\pi\epsilon^2}2[g^{\rho\mu}(1+O(\epsilon^2)) + \varepsilon^{\rho\mu}\sqrt{-\lambda a}\epsilon^\sigma\partial_\sigma\phi_2(x) + O(\epsilon^2)]$$

$$= \frac{1}{2\pi}(\sqrt{-\lambda a}\partial_\rho\phi_1(x)g^{\rho\mu} - \varepsilon^{\rho\mu}\sqrt{-\lambda\tilde{a}}\partial_\rho\phi_2(x)) = \frac{\lambda}{2\pi}j^\mu_+(x)$$

□

References

[1] Johnson, K. *Solution of the Equations for the Green's Functions of a two Dimensional Relativistic Field Theory*, Il Nuovo Cimento **20**(1961) pp.773-790

[2] Klaiber, B. *The Thirring model*, In: *Quantum theory and statistical physics*, Editors: Barut, Brittin. Gordon and breach, New York, 1968. (Lectures in theoretical physics; Vol **10-A**). pp. 141-176

[3] Knizhnik, V.G.;Zamolodchikov, A.B. *Current algebra and Wess-Zumino model in two dimensions*, Nuclear Physics **B247**(1984) pp. 83-103

[4] de Mirleau, O. *Non-Gaussian generalizations of Wick's theorems, related to the Schwinger-Dyson equation.* hep-th/9507039, *Combinatorical Aspects of the Schwinger-Dyson Equation*, Journal of Geometry and Physics **21**(1997) pp. 357-380

[5] de Mirleau, O. *Concerning a natural compatibility condition between the action and the renormalized operator product*, hep-th/9608120

[6] Thirring, W. *A Soluble Relativistic Field Theory*, Annals of Physics **3**(1958) pp.91-112

[7] Wightman, A. *Models in Two-Dimensional Space-Time*, Chapter 4 of *Introduction to some Aspects of Quantized Fields.* In: Lecture notes, Cargese summer school, 1964, Gordon and Breach, New York, pp. 196-240.

Department of Mathematics, University of Amsterdam, Plantage Muidergracht 24, NL-1018 TV Amsterdam, The Netherlands, e-mail: mirleau@wins.uva.nl